EFFECTIVE RECONSTRUCTION OF CURVES FROM THEIR THETA HYPERPLANES

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Abstract. Working over the complex numbers, and considering a generic curve C of genus $g > 3$ and a non-trivial 2-torsion point α on it's Jacobian JC, we give a new interpretation of Mumfords map $\chi_- : H^0(JC, \Theta_C + t_\alpha \Theta_C)_- \overset{\cong}{\to} H^0(\mathrm{Prym}(C,\alpha), 2\Theta_{\mathrm{Prym}(C,\alpha)})$. We use this interpretation to *effectively* reconstruct C from its odd theta hyperplanes; hence we also get an effective generic-inverse for the Torelli map.

1. Introduction

Throughout this paper we work over the complex numbers. We consider a generic curve C of genus $g \geq 3$, a non-trivial 2-torsion point α on the Jacobian of C – denoted JC – and the corresponding double cover $\tilde{C}/C$. We present two results: The first concerns one of Mumfords isomorphisms from [Mu] p. 335 (see the introduction in [IzPa] for a more detailed exposition than Mumfords):

$$\chi_- : H^0(JC, \Theta_C + t_\alpha \Theta_C)_- \overset{\cong}{\to} H^0(P, 2\Xi),$$

where (P, Ξ) is the Prym variety of (C, α), where t_α is the translation by α, and where the subscript $_-$ denotes the t_α anti-invariant eigen-space. We construct a natural inclusion:

$$\Psi : \wedge^2 H^0(C, \omega_C) \to H^0(JC, \Theta_C + t_\alpha \Theta_C)_{0-},$$

where the subscript $_0$ means the sections which are trivial at the 0 of the respective Abelian variety. Denoting by $\mathfrak{T}_2$ the map taking a section to its second Taylor terms, we prove the following:

1.1. **Theorem.** *The composition of maps*

$$\mathfrak{T}_2 \circ \chi_- \circ \Psi : \wedge^2 H^0(C, \omega_C) \to S^2 H^0(C, \omega_C + \alpha),$$

is an isomorphism.

The second result is an effective reconstruction algorithm: Starting from the theta hyperplanes associated with the theta characteristics in the Steiner system Σ of the pair (C, α) (the set of theta characteristics such that both θ and $\theta + \alpha$ are odd), we construct a finite number of quadrics in $|\mathcal{I}_{\tilde{C}}(2)|$, whose intersection is the canonical image of $\tilde{C}$. Since we have an algorithm,

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we present it as a programme – to be carried out – and not as one big theorem:

(1) In theorem 3.2 we construct the image of the odd Prym theta hyperplanes in $|\omega_C + \alpha|$ from the images of the Steiner system Σ in $\mathbb{P}\wedge^2 H^0(C, \omega_C)$ (in genus 3 this is part of the classical Coble-Recillas construction - see [Co] section 49, in genus > 3 we use theorem 1.1 and its consequences).
(2) In theorem 3.3 we reconstruct the images of pairs of points in the Steiner system Σ inside $\mathbb{P}S^2H^0(\tilde{C}, \omega_{\tilde{C}})$ from their images in $\mathbb{P}S^2H^0(C, \omega_C)$ and $\mathbb{P}H^0(C, \omega_C + \alpha)$.
(3) In theorem 3.7 we show that the intersection of these quadrics (the images of the points of Σ in $\mathbb{P}S^2H^0(\tilde{C}, \omega_{\tilde{C}})$) is indeed $\tilde{C}$.

Our result transforms the main result of [CaSe] into an effective one. Moreover, we get an effective generic inverse Torelli: one starts with an explicit presentation of JC and all the 2-torsion points, and use this data to compute the intersection of the quadrics q_i for some chosen α. Generically one is guarantied to get $\tilde{C}$; from which one easily gets the pair (C, α) (note that in [De], [LaSe] the authors compute the Prym canonical curve – for a generic curve – as an intersection of certain quadrics, but the quadrics they use cannot be described in an effective manner).

2. A natural isomorphism $\wedge^2 H^0(C, \omega_C) = S^2H^0(C, \omega_C + \alpha)$

2.1. Up until the proof of theorem 1.1 (which immediately follows 2.9) we consider only smooth curves. We outline the definitions in 2.2. From 2.11 onwards we also consider certain nodal curves with specific level structures. The relevant definitions, and extensions of the definitions given for smooth curves are given in 2.11.

2.2. **Definition.** As explained in the introduction, throughout the paper we consider an unramified double cover $\tilde{C}/C$ where C is a curve of genus $g \geq 3$, and where the double cover corresponds to a non-trivial 2-torsion point α in the Jacobian of C. We denote the respective Jacobians by $JC, J\tilde{C}$, and the respective theta divisors by $\Theta_C, \Theta_{\tilde{C}}$.

We denote the Prym variety of the pair (C, α) by P, and its theta divisor by Ξ (assumed to be symmetric and through 0). We denote the Steiner system corresponding to α by Σ; note that the map $JC \to J\tilde{C}$ sends pairs of odd theta characteristics in Σ to the odd theta characteristics of P; specifically these images are all singular points of $\Theta_{\tilde{C}}$.

We denote the projections coming from the direct sum presentation $H^0(\tilde{C}, \omega_{\tilde{C}}) \cong H^0(C, \omega_C) \oplus H^0(C, \omega_C + \alpha)$ by p_1, p_2; we make the two standard identifications $T_0JC = H^0(C, \omega_C)$ and $T_0P = H^0(C, \omega_C + \alpha)$.

We denote by Ver_n the degree n Veronese map taking the projective space spanned by $x_0, \ldots x_d$ to the projective space spanned by the degree n monomials in $x_0, \ldots x_d$.

We denote the subscheme of singular (respectively smooth) points of a scheme X by X^{sing} (respectively X^{smooth}). Let A be an Abelian variety, then for $x \in \Theta_A^{\mathrm{smooth}}$ we the denote the tangent plane to Θ_A at x inside T_xA by $T_x\Theta$; whereas for $x \in \Theta_A^{\mathrm{sing}}$ we denote the (quadric) tangent cone to Θ_A at x inside T_xA by $\mathcal{T}_x\Theta_A^{\mathrm{sing}} \subset T_xA$. We always use the standard identification $T_aA = T_0A$ for points $a \in A$

Finally we denote by Σ^0 the set of points $x \in \Sigma$ such that $x, t_\alpha x$ are smooth points on Θ, and such that the image of x in Ξ is smooth; note that for generic curves $\Sigma^0 = \Sigma$, because generic curves do not have a theta null.

2.3. **Proposition.** *The diagram:*

$$
\begin{tikzcd}
\Sigma^0 \arrow[rr, "T_*\Theta_C \otimes T_{*+\alpha}\Theta_C"] \arrow[dd, "/\alpha"] & & \mathbb{P} \otimes^2 H^0(C,\omega_C) \arrow[r, dashed] \arrow[dr, dashed] & \mathbb{P} \wedge^2 H^0(C,\omega_C) \\
 & & & \mathbb{P}S^2H^0(C,\omega_C) \\
\Sigma^0/\alpha \arrow[r] \arrow[dr] & \Theta_{\tilde{C}}^{sing} \arrow[r, "\mathcal{T}_*\Theta_{\tilde{C}}"] & \mathbb{P}S^2H^0(\tilde{C},\omega_{\tilde{C}}) \arrow[ur, dashed, "\mathbb{P}\circ S^2p_1"] \arrow[dr, dashed, "\mathbb{P}\circ S^2p_2"] & \\
 & \Xi^{smooth} \arrow[u] \arrow[r, "T_*\Xi"] & |\omega_C + \alpha| \arrow[r, "\mathrm{Ver}_2"] & \mathbb{P}S^2H^0(C,\omega_C+\alpha)
\end{tikzcd} \tag{1}
$$

commutes.

Proof. Diagram chasing. □

In the rest of this section we will follow our reconstruction programme, and "add" to diagram 1 (for generic curves, as well as certain nodal curves) an isomorphism between $\mathbb{P}\wedge^2 H^0(C,\omega_C)$ and $\mathbb{P}S^2H^0(C,\omega_C+\alpha)$. We start by investigating a special case, where C is hyperelliptic and α is the difference between two Weierstrass point.

2.4. **Proposition.** *Assume that C is hyperelliptic curve with Weierstrass points $w_1,\dots,w_{2g+2}$, and recall that the canonical image of C is a rational normal curve. Denote by x_i the canonical image of the Weierstrass w_i. Then the theta hyperplanes corresponding of C are hyperplanes which intersect the canonical curve on $g-1$ of the x_is (counting with multiplicities). The non theta null ones are the ones where the x_i are all distinct.*

Proof. Classical. □

2.5. **Proposition.** *Let C, w_i be as in 2.4, let $\alpha = w_{2g+1} - w_{2g+2}$, let $a_1,\dots,a_{2g+2} \in \mathbb{P}^1$ be the images of the Weierstrass points of C under the hyperelliptic cover, and let H be the hyperelliptic curve ramified over $a_1,\dots,a_{2g}$. Then the Prym variety of (C,α) is JH.*

Proof. See [Do] example 2.10.iii. □

2.6. **Lemma.** *Let C, w_i, x_i, a_i, α be as in 2.4, 2.5, then Σ^0 is the set of* all *pairs of hyperplanes whose intersections with the canonical image of C are given by*

$$2(x_{2g+1} + \sum_{i\in I} x_i), \quad 2(x_{2g+2} + \sum_{i\in I} x_i),$$

where $I \subset \{1, \dots, 2g\}, \# I = g-2$. Moreover, the image of Σ^0 in $\mathbb{P}\wedge^2 H^0(C, \omega_C)$ is given by $\{\pm \wedge_{i\in I} x_i\}_{\substack{I\subset\{1,\dots,2g\}\\ \# I = g-2}}$,

Proof. By 2.4, a pair of hyperplanes corresponding to two non singular theta characteristics on Θ_C with difference α intersect the canonical image of C as prescribed in the statement of the lemma.

Such a pair of theta characteristics maps to some odd theta characteristic of H, which by 2.5 is the sum of the Weierstrass points of H projecting to $\{a_i\}_{i\in I}$ (under the hyperelliptic map on H). However, by 2.4 this theta characteristic is a non singular 2-torsion point on Ξ. Hence this pair lies inside Σ^0, and the result follows. $\square$

2.7. **Proposition.** *For a generic curve C the image of Σ inside $\mathbb{P}\wedge^2 H^0(C, \omega_C)$ in diagram 1 spans the space.*

Proof. Let C be a generic hyperelliptic curve, α a difference of two Weierstrass points, and use the notations of 2.4, then the x_is are in general position in $|\omega_C| \cong \mathbb{P}^{g-1}$. The image of Σ^0 in $\mathbb{P}\wedge^2 H^0(C, \omega_C)$ is given by

$$\left\{ T_x\Theta_{\tilde{C}} \wedge T_{x+\alpha}\Theta_{\tilde{C}} | x \in \Sigma^0 \right\} \supset \{\pm \wedge_{i\in I} x_i\}_{\substack{I\subset\{1,\dots,2g\}\\ \# I = g-2}},$$

which contains the spanning set $\{\pm \wedge_{\substack{k\le g+1\\ k\ne i,j}} x_k\}_{i\ne j}$. $\square$

2.8. **Proposition.** *For a generic curve C, the image of Σ inside $\mathbb{P}S^2H^0(C, \omega_C + \alpha)$ in diagram 1 spans the space.*

Proof. It suffices to proof the claim where C is a generic hyperelliptic curve, and α is a difference between two Weierstrass points. In this case the claim follows from the description of Σ^0 in 2.6, the fact that by 2.5 the Prym of C, α is Hyperelliptic with generic Weierstrass points, and the description of the theta hyperplanes in 2.4 - this time applied to the Prym variety. $\square$

2.9. **Proposition-Definition** (Definition of the map Ψ)**.** Denote by L the sheaf $\Theta_C + t_\alpha\Theta_C$ on JC, then we have an exact sequence of sheaves:

$$0 \to \mathcal{O}_{JC}(t_\alpha\Theta_C) \to \mathcal{O}_{JC}(L) \to \mathcal{O}_{\Theta_C}(L) \to 0.$$

Since both $t_\alpha\Theta$ and L are ample, the only non trivial cohomologies of $\mathcal{O}_{JC}(\Theta_C + \alpha), \mathcal{O}_{JC}(L)$ are the 0th. Hence we have an natural identification:

$$H^0(\Theta_C, L) \cong \mathrm{coker}(H^0(JC, t_\alpha\Theta_C) \to H^0(JC, L)).$$

However, since $H^0(JC, t_\alpha\Theta) \cong \mathbb{C}$, we get a natural isomorphism

$$H^0(\Theta_C, L) \cong H^0(JC, L)_0.$$

This identification immediately gives us a map

$$H^0(\Theta_C, \mathcal{O}_{\Theta_C}(\Theta_C)) \otimes H^0(\Theta_C, \mathcal{O}_{\Theta_C}(t_\alpha \Theta_C)) \to H^0(\Theta_C, L) \cong H^0(JC, L)_0.$$

Moreover, since the sequence

$$0 \to \mathcal{O}_{JC} \to \mathcal{O}_{JC}(\Theta_C) \to \mathcal{O}_{\Theta_C}(\Theta_C) \to 0$$

is exact, we get a natural isomorphism $H^0(\Theta_C, \mathcal{O}_{\Theta_C}(\Theta_C)) \cong H^1(\mathcal{O}_{JC})$. Using this isomorphism, and the analogous one for $H^0(t_\alpha\Theta_C, \mathcal{O}_{t_\alpha\Theta_C}(t_\alpha\Theta_C))$, we get a map

$$H^1(\mathcal{O}_{JC}) \otimes H^1(\mathcal{O}_{JC}) \to H^0(JC, L)_0,$$

where the first coordinate should be thought of as tangents to theta divisor, and the second with tangents to the shifted theta divisor. It remains to find the t_α anti-invariant eigen-space of $H^1(\mathcal{O}_{JC}) \otimes H^1(\mathcal{O}_{JC})$, which is clearly $H^1(\mathcal{O}_{JC}) \wedge H^1(\mathcal{O}_{JC})$.

Proof of Theorem 1.1. The image of Σ^0 in $\mathbb{P} \wedge^2 H^0(C, \omega_C)$ – in diagram 1 – maps to it's image in $\mathbb{P}H^0(JC, L)_{0-}$. Moreover, the image of Σ^0 in $\mathbb{P}^2 S^2 H^0(C, \omega_C + \alpha)$ – again in diagram 1 – is the second Taylor coefficients of it's image in $\mathbb{P}(P, \Xi_0)$. Since both images span the respective spaces, our claim holds. □

2.10. **Notation.** Henceforth, we denote the isomorphism

$$\wedge^2 H^0(C, \omega_C) \to S^2 H^0(C, \omega_C + \alpha)$$

by Φ, and the induced map on the projectivized spaces by ϕ.

2.11. **Proposition-Definition.** Let C be an irreducible nodal curve with a simple node p (and possibly other simple nodes), let C' be it's normalization at p, let C'' be it's full normalization, and let $\tilde{C}''$ be an unramified double cover of C''. Let $\tilde{C}, \tilde{C}'$ be the pullbacks of $\tilde{C}$ under the map $C \to C''$, $C' \to C''$. Then the following properties are classical and well known:

(1) The dual linear system $|\omega_C|^*$ projects to the dual linear system $|\omega_{C'}|^*$ from the canonical image of p (where we denote by ω the dualizing sheaves of the respective curve).
(2) The Steiner system of associated with the double cover $\tilde{C}'' \to C''$ pulls back under $C \to C''$ to a set of generalized theta hyperplanes through the canonical image of p.
(3) Denote the two preimages of p in $\tilde{C}$ by p_1, p_2, then the dual linear system $|\omega_{\tilde{C}}|^*$ projects to the dual linear system $|\omega_{\tilde{C}'}|^*$ from the line connecting the canonical images of p_1, p_2.
(4) By 1 and 3, we have a projection

$$\mathbb{P}(H^0(\tilde{C}, \omega_{\tilde{C}})^* / H^0(C, \omega_C)^*) \to \mathbb{P}(H^0(\tilde{C}', \omega_{\tilde{C}'})^* / H^0(C', \omega_{C'})^*),$$

from the quotient of the pullback of the line $\overline{p_1 p_2}$ by the pullback of the canonical image of p.

Moreover, by [AlBiHu] 5.2.1 the Prym varieties of $\tilde{C} \to C$ and $\tilde{C}' \to \tilde{C}$, which we denote by P, P' are well defined semi-Abelian varieties, and $P \to P'$ is a $\mathbb{C}^*$ torsor.

2.12. **Proposition-Definition.** Let $\tilde{C} \to C$ be an admissible double cover in $\overline{\mathcal{R}_g}$ such that the images of the generalized Steiner system (and it's quotient on the generalized Prym variety) span the spaces $\mathbb{P} \wedge^2 H^0(C, \omega_C)$ and $\mathbb{P}S^2((H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(C, \omega_C))$ respectively. Note that since generalized theta hyperplanes are continues on families, so are the maps $\Sigma^0 \to \mathbb{P} \wedge^2 H^0(C, \omega_C)$ and $\Xi^{\text{smooth}} \overset{\text{Ver}_2 \circ T_*}{\to} \mathbb{P}S^2((H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(C, \omega_C))$ (which for smooth curves appear in diagram 1).

We claim that we may extend our definition of ϕ to the linear map

$$\phi : \mathbb{P} \wedge^2 H^0(C, \omega_C) \to \mathbb{P}S^2(H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(C, \omega_C)),$$

which sends the image of Σ^0 to the image of the quotient of Σ^0 by the involution. Moreover, we claim that our extension of ϕ commutes with the maps from Σ^0 to $\mathbb{P} \wedge^2 H^0(C, \omega_C)$ and $\mathbb{P}S^2((H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(C, \omega_C))$, and is continuous on families.

Proof. For a generic curve of genus ≥ 3, theorem 1.1 shows that there is an isomorphism between $\mathbb{P} \wedge^2 H^0(C, \omega_C)$ and $\mathbb{P}S^2 H^0((H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(C, \omega_C))$ that commutes with the morphisms in diagram 1.

The map is defined on the domain we claimed since the images of the Steiner system in the spaces $\mathbb{P} \wedge^2 H^0(C, \omega_C)$ and $\mathbb{P}S^2((H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(C, \omega_C))$ are continuous on families, and since – by our assumption – they span them.

The map is continuous on families since the images of the Steiner system in the above spaces are continuous on families. $\square$

2.13. **Lemma.** *With the notations of 2.11, and assuming the map ϕ is defined on both $\tilde{C}/C$ and $\tilde{C}'/C'$, the following diagram commutes:*

$$\begin{array}{ccc}
\mathbb{P} \wedge^2 H^0(C, \omega_C) & \xrightarrow{\phi} & \mathbb{P}S^2((H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(C, \omega_C)) \\
\uparrow & & \uparrow \\
\mathbb{P} \wedge^2 H^0(C, \omega_{C'}) & \xrightarrow{\phi} & \mathbb{P}S^2((H^0(\tilde{C}', \omega_{\tilde{C}'})/H^0(C', \omega_{C'})),
\end{array}$$

where the projections are the ones from 2.11.

Proof. Our claim follows from the continuity of Steiner systems and the morphisms in diagram 1 on families, and from 2.11 $\square$

2.14. **Corollary.** *The map ϕ is defined on generic irreducible $\tilde{C} \to C$ such that C has $k \leq g-1$ nodes, and where $\tilde{C}$ is a pullback of an admissible double cover of the normalization of C.*

Proof. We consider a degenerated hyperelliptic curve C with nodes $n_1, \ldots n_k$, and Weierstrass points $w_1, \ldots w_m$ (where $m = 2g + 2 - 2k$). We consider C

as a limit of hyperelliptic curves, with k pairs of Weierstrass points which approach each other along a family.

By 2.11 (specifically, the part we quote from [AlBiHu] 5.2.1), the Prym construction is continues along families approaching $\tilde{C} \to C$. Hence 2.4, 2.5, and 2.6 all hold in our degenerated case, where we take nodes and Weierstrass points instead of Weierstrass points only. Moreover, as long as $m-2+k \geq g+1$, the proofs of 2.7 and 2.8 although carry verbatim to our case. $\square$

3. A generic inverse Torelli

3.1. Following our programme, we now have to reconstruct the Ver_2 image of $|\omega_C+\alpha|$ inside $\mathbb{P}S^2H^0(C,\omega_c+\alpha)$ from the image of Σ^0/α in $\mathbb{P}S^2H^0(C,\omega_C+\alpha)$. Below we will discuss both smooth and nodal curves; however, we will only use the notation α if the curve involves is smooth.

3.2. **Theorem.** *With our standard notations, and where C is a generic smooth curve, the image of*

$$\mathrm{Ver}_2 : |\omega_C+\alpha| \to \mathbb{P}S^2H^0(C,\omega_C+\alpha)$$

is the intersection of the quadrics containing all the images of the odd theta hyperplanes of P in $|\omega_C+\alpha|$.

Proof. Since we seek a generic result, it suffices to produce one example where the proposition holds. W.l.o.g. the curve C is hyperelliptic, and $P = JH$, where H is a generic hyperelliptic curve of genus $g-1$. Let $x_1,\dots x_{2g}$ be the images of the Weierstrass points of H in $|\omega_H|^* = |\omega_C+\alpha|^*$, then by 2.5 and 2.6 the theta hyperplanes corresponding to the image of Σ^0 in $|\omega_H|$ are hyperplanes W such that $W\cdot H$ is a sum of $g-2$ distinct x_i's.

We observe that it suffices to show that

$$\text{span } S^2\left\{(\wedge_{i\in I}x_i)^{\otimes 2}\right\}_{\substack{I\subset\{1,\dots,2g\}\\ \#I=g-2}} = \text{span } S^2\left\{v^{\otimes 2}\right\}_{v\in|\omega_H|}. \tag{2}$$

Indeed, such an equality would mean that the only quadric forms on $\mathbb{P}S^2H^0(C,\omega_C+\alpha)$ that are trivial on all the images of the odd theta hyperplanes in $\mathbb{P}S^2H^0(H,\omega_H)$, are trivial on the entire image of $\mathrm{Ver}_2(|\omega_H|)$.

To prove that equation 2 holds for a generic H, we recall that there is a rational normal curve through any $g+1$ points in $\mathbb{P}^{g-2}$. Hence, we choose Weierstrass points such that $x_1,\dots x_g$ form a basis of $|\omega_H|^*$, and such that $x_{g+1} = \sum_{i=1}^{g} x_i$. For $i\leq g$ we denote $\hat{x}_i := \wedge_{\substack{j\leq g\\ j\neq i}} x_j$. We now observe that:

$$v_i := \bigwedge_{\substack{j\leq g\\ j\neq i}} x_j = \hat{x}_i,$$

$$v_{ij} := x_{g+1}\wedge \bigwedge_{\substack{j\leq g\\ j\neq i,k}} x_j = \pm\hat{x}_i \pm \hat{x}_k \quad \text{for } i\neq j.$$

We now get:

$$v_i^{\otimes 2} \otimes v_j^{\otimes 2} = \hat{x_i}^{\otimes 2} \otimes \hat{x_j}^{\otimes 2},$$

where as for $j \neq k$

$$\begin{aligned} v_i^{\otimes 2} \otimes v_{jk}^{\otimes 2} =& \hat{x_i}^{\otimes 2} \otimes (\hat{x_j} \pm \hat{x_k})^{\otimes 2} \quad \Rightarrow \\ v_i^{\otimes 2} \otimes v_{jk}^{\otimes 2} -& v_i^{\otimes 2} \otimes (v_j^{\otimes 2} + v_k^{\otimes 2}) = \pm 2\hat{x_i}^{\otimes 2} \otimes (\hat{x_j} \otimes \hat{x_k}), \end{aligned}$$

and finally for the case $i \neq j, k \neq l$ we get:

$$\begin{aligned} v_{ij}^{\otimes 2} \otimes v_{kl}^{\otimes 2} =& (\hat{x_i} \pm \hat{x_j})^{\otimes 2} \otimes (\hat{x_k} \pm \hat{x_l})^{\otimes 2} \quad \Rightarrow \\ & v_{ij}^{\otimes 2} \otimes v_{kl}^{\otimes 2} - (v_i^{\otimes 2} + v_j^{\otimes 2}) \otimes v_{kl}^{\otimes 2} - v_{ij}^{\otimes 2} \otimes (v_k^{\otimes 2} + v_l^{\otimes 2}) \\ & + 2(v_i^{\otimes 2} + v_j^{\otimes 2}) \otimes (v_k^{\otimes 2} + v_l^{\otimes 2}) = \pm 4(\hat{x_i} \otimes \hat{x_j}) \otimes (\hat{x_k} \otimes \hat{x_l}), \end{aligned}$$

and the proposition follows. □

3.3. **Theorem.** *For a generic pair (C, α), where C is smooth, and a point $x \in \Sigma^0/\alpha \in \Theta_{\tilde{C}} \subset J\tilde{C}$, the tangent cone to $x \in \Theta_{\tilde{C}}^{sing}$ lies in $\mathbb{P}(S^2H^0(C, \omega_C) \oplus S^2H^0(C, \omega_C + \alpha))$.*

Proof. We consider the eigen-spaces of $S^2H^0(\tilde{C}, \omega_{\tilde{C}})$ under the action of the involution on $\tilde{C}$: Since the fixed and anti-fixed eigen-spaces of $H^0(\tilde{C}, \omega_{\tilde{C}})$ are $H^0(C, \omega_C)$ and $H^0(C, \omega_C + \alpha)$ respectively, the fixed and anti-fixed eigen-spaces of $S^2H^0(\tilde{C}, \omega_{\tilde{C}})$ are

$$S^2(H^0(C, \omega_C)) \oplus S^2(H^0(C, \omega_C + \alpha)) \text{ and } H^0(C, \omega_C) \otimes H^0(C, \omega_C + \alpha)$$

respectively.

Since the point x is invariant under the shift by α, it's tangent cone lies either in the fixed or in the anti-fixed eigen space of the shift by α involution. However, we already know that the tangent cone is non trivial on the fixed part; hence it lies on the fixed part. □

3.4. **Definition.** Let (C, α) be as above, let l_i, l_i' be to odd theta hyperplanes of C, such that the difference between the corresponding theta characteristics is α. Let L_i, L_i' be some corresponding linear forms. Let $M_i \in S^2H^0(C, \omega_C + \alpha)$ be the linear form such that $M_i^2 = \Phi(L_i \wedge L_i')$ (which – by theorem 3.2 we may reconstruct from all the l_i, l_i's up to a linear change of coordinates on $H^0(C, \omega_C + \alpha)$), then we define the quadric form (over $H^0(\tilde{C}, \omega_{\tilde{C}}) \cong H^0(C, \omega_C) \oplus H^0(C, \omega_C + \alpha)$)

$$Q_i := L_i L_i' + M_i^2.$$

We denote by q_i the respective null set (had we chosen other linear forms whose null sets are still l_i, l_i', the quadric Q_i would only be changed by a scalar).

By 2.11 and 2.14 we may extend this definition to the case where the curve C is irreducible and at most nodal with $g-1$ nodes, and where $\tilde{C} \to C$ is a pullback of some none trivial admissible double cover of the normalization of C.

3.5. **Proposition.** *With our usual notations, for any pair of points $x_i, x_i+\alpha$ in Σ^0, the corresponding quadric q_i is in fact the tangent cone*

$$\mathcal{T}_{x_i}\Theta_{\tilde{C}} \in \mathbb{P}S^2H^0(\tilde{C},\omega_{\tilde{C}}).$$

Proof. This follows from theorems 3.2 and 3.3. □

3.6**.** As we just saw, we can recover from the image of the points of Σ in $H^0(C,\omega_C)$ many quadrics of the form $\mathbb{P}\mathcal{T}_x\tilde{C}$ where $x \in \tilde{C}^{\text{sing}}$. Recall (see [ACGH] theorem VI.4.1), that the intersection of all these quadrics is the canonical image of $\tilde{C}$; the question is if we now have enough such quadrics so that their intersection would be $\tilde{C}$, this is the last step in our programme, and we answer it affirmatively - for a generic curve - in the theorem below.

3.7. **Theorem.** *Let $\tilde{C}/C$ be an admissible double cover of a smooth curve, where $g \geq 3$, then the canonical image of $\tilde{C}$ is the intersection of the quadric forms q_i associated to the points $x_i \in \Sigma$.*

Proof. We prove the proposition by inducting on the genus g, starting with the classical case $g = 3$. In this case $\dim(I_2(\tilde{C})) = 2$. Hence, it suffices to show that there exists three non-collinear (in the space of quadrics) q_i's. To do this we note that the quadratic forms L_iL_i' and $M_i^{\otimes 2}$ are supported on different factors in the direct sum

$$S^2H^0(\omega_C) \oplus S^2((H^0(\tilde{C},\omega_{\tilde{C}})/H^0(C,\omega_C)),$$

so it suffices to prove that the six quadric forms L_iL_i' span a 3-dimensional family of quadrics forms in $H^0(|\omega_C|,\mathcal{O}_{|\omega_C|}(2))$ for a generic curve C. Indeed, for a general curve C, the singularities of the six conics $l_i \cup l_i'$ lie on a general conic, so the the corresponding quadric forms cannot be collinear; thus the proposition holds for $g = 3$.

We now turn to the general case. Let C_0 be a generic irreducible k-nodal curve, with nodes $x_1, \ldots, x_k$, let $\tilde{C}_0/C_0$ be an admissible double cover which is a pullback from the normalization of C_0. Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_k$ be generic curves in $\Delta_0 \subset \overline{\mathcal{R}_g}$, which contain $[\tilde{C}_0/C_0]$, and such that each of them "preserves" a different node of C_0. Denote by $\mathrm{Def}_i(C_0)$ the deformation of C_0 along $\mathfrak{X}_i$. Finally, since we are proving a generic result, we may assume w.l.o.g that $\tilde{C}/C$ is a deformation of $\tilde{C}_0/C_0$.

Let $\tilde{C}_i'/C_i'$ be the admissible double cover lying over a generic moduli point in on of the $\mathfrak{X}_i$s, and let $\tilde{C}_i''/C_i''$ be it's normalization. By 2.13, the quadrics corresponding to $\tilde{C}_i''/C_i''$ (in the sense of 3.4) over the dual linear system $|\omega_{\tilde{C}_i''}|^*$ pull back under the projection from 2.13 to a subset of the quadrics corresponding to $\tilde{C}_i'/C_i'$ over $|\omega_{\tilde{C}_i'}|^*$ (again, in the sense of 3.4). We denote this projection by π_i.

Note that $\tilde{C}_i'$ lies in the inverse image under π_i of the intersection of the quadrics corresponding to $\tilde{C}_i''/C_i''$, which by the induction hypothesis is $\pi_i^*\tilde{C}_i''$.

Identifying all the complete linear systems of the dualizing sheaves of deformations of C_0 with that of C_0 (via the deformations of the curves, which induce deformations of the linear systems), we conclude that the intersection of quadrics corresponding to $\tilde{C}/C$ is a subscheme of

$$\cap_{i\leq k}\mathrm{Def}'_i(\pi_i^*(\pi_i(\mathrm{Def}_i(C_0)))),$$

where Def'_i is some deformation in the space of quadric forms over $|\omega_{C_0}|^*$. However, this intersection is equal to the intersection

$$\cap_{i\leq k}\mathrm{Def}''_i(\pi_i^*(\pi_i(C_0))), \tag{3}$$

where Def''_i is some deformation in the space of quadric forms over $|\omega_{C_0}|^*$. Finally note that by 2.11.3 the cones we get by pulling curve back under π_i^* have $\mathbb{P}^2$ fibers, each spanned by some point, and the line connecting the two corresponding nodes. Since the lines connecting the pairs of nodes of $\tilde{C}_0$ are in general linear position, the intersection in equation 3 is contained in some deformation of $\tilde{C}_0$ if $k\geq 3$.

Since the intersection of these quadrics certainly contains $\tilde{C}$ (they are all in $I_2(\tilde{C})$ - see [ACGH] VI.4.1), we are done. □

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